Oxford Cambridge and RSA

Monday 19 October 2020 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension Insert

Time allowed: 2 hours

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INSTRUCTIONS
- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION
- This Insert contains the article for Section B.
- This document has **4** pages.

Which is bigger?

Which is bigger: π^{e} or e^{π}? Using a calculator confirms that e^{π} is the larger, but how can this be proved without the use of a calculator?

Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that $3^4 > 4^3$. In the expression 3^4, 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where $a^b > b^a$ if $a < b$. That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that $a^b > b^a$ if $a < b$ and both a and b are integers greater than 1. However, it is also possible to find counter examples to this conjecture.

Exponents can also be rational numbers, and in general $x^{\frac{p}{q}}$ denotes $\left(\sqrt[q]{x}\right)^p$ where p and q are integers and q is positive. So, any rational power of a positive number, x, can be defined. However, both e and π are irrational numbers. Considering the original question about π^{e} and e^{π} raises the issue of what is meant by an irrational power of a number.

Extending the definition of power to irrational numbers

What, for example, is meant by 2^{π}?

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to π.

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, …

Using a spreadsheet gives a sequence of approximations to 2^{π}, as shown in Fig. C1. The limit of this sequence of approximations is the value of 2^{π}. This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy.

	A	B
1	k	2^k
2	3	8
3	3.1	8.574188
4	3.14	8.815241
5	3.142	8.82747
6	3.1416	8.825023
7	3.14159	8.824962

Fig. C1

2^x and x^2 are increasing functions of x for $x > 0$ and this allows us to deduce that $\pi^2 > 2^{\pi}$, as follows.

We know that π is between 3 and 3.142

$\pi < 3.142 \Rightarrow 2^{\pi} < 2^{3.142} = 8.82747$

$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$

So $\pi^2 > 9 > 8.82747 > 2^{\pi}$

Hence $\pi^2 > 2^{\pi}$

Which is bigger: π^{e} or e^{π}?

An indirect method, using calculus, enables us to prove that e^{π} is larger than π^{e}. Fig. C2 shows the curve $y = \frac{1}{x}$ in the first quadrant together with the rectangle with vertices at the points (e, 0), $\left(\mathrm{e}, \frac{1}{\mathrm{e}}\right)$, $\left(\pi, \frac{1}{\mathrm{e}}\right)$ and (π, 0). We use the fact that the area under the curve between e and π is less than the area of this rectangle.

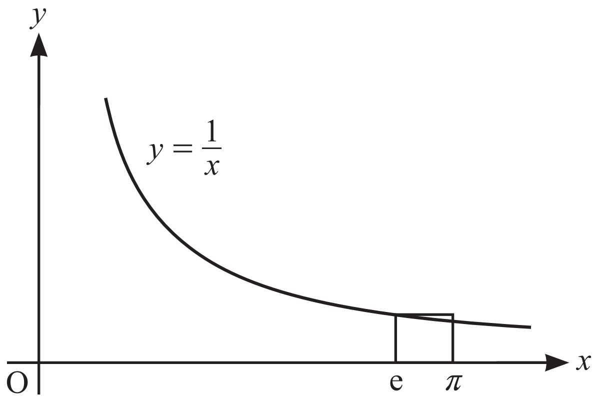

Fig. C2

The area of the rectangle is $\frac{1}{\mathrm{e}}(\pi - \mathrm{e})$

$$\int_{\mathrm{e}}^{\pi} \frac{1}{x}\,\mathrm{d}x < \frac{1}{\mathrm{e}}(\pi - \mathrm{e})$$

$$\ln \pi - 1 < \frac{\pi}{\mathrm{e}} - 1$$

$$\ln \pi < \frac{\pi}{\mathrm{e}}$$

e^x is an increasing function for all values of x

hence $\pi < \mathrm{e}^{\frac{\pi}{\mathrm{e}}}$

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power e gives the desired result.

Using a similar method, it can be shown that $\mathrm{e}^a > a^{\mathrm{e}}$ for any positive number $a \neq \mathrm{e}$.

An alternative method for showing that $\mathrm{e}^a > a^{\mathrm{e}}$ for any positive number a is to show that the only stationary point on the curve $y = \frac{\ln x}{x}$ (a maximum) occurs where $x = \mathrm{e}$.